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Abstracts from the International Conference on Mathematical Problems from the Physics of Fluids

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INTRODUCTION

We present here the abstracts of the talks presented at the conference "Mathematical Problems from the Physics of Fluids." Each invited speaker was requested to provide a short abstract of his talk together with a commented bibliography containing the papers of the last decades judged (by the speaker) as particularly relevant for his work and/or his talk.

Unfortunately, as the reader can check, not all the speakers interpreted our request in the sense we wanted: nevertheless we think that all the contributions are interesting and important even when they are a little too sketchy. Their collection will hopefully be useful to those who would have liked to attend the conference but could not come.

The aim of the conference was to bring together scientists actively working on the problems of turbulence and of related chaotic phenomena, the idea being to provide an updated review of the research performed theoretically, numerically, and experimentally on the above subjects.

In recent years we have witnessed the exciting experience of new theories and viewpoints developing simultaneously and with various mutual influences in the mathematical theory of dynamical systems, in the theory of fluids and in actual experiments on fluids. This conference tried to reproduce for a short time the atmosphere which characterized the last decade of research on fluids: the results have been quite interesting. We are quite confident that the mutual collaboration and interaction between people with very diverse cultural backgrounds (e.g., mathematicians and

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experimental physicists) will continue despite some of the talks hinting that this happy period might have reached an end.

The latter pessimistic view arises from the feeling among some participants that on one hand we have basically understood the onset of turbulence to an extent that it might seem pointless to pursue further experiments to check agreement between theory and experiment and on the other hand that theorists seem unable to get a simple clue to the interpretation of the more developed turbulence and its intricate space-time patterns. Thus, while the experiental techniques are so advanced as to allow the performance of very elaborate experiments, the theorists, unable to build a really solid theory of the statistical properties of turbulent flows, seem to indulge in trying to understand better the onset of turbulence.

Of course one has to agree that we are not close to understanding the structure of the strange attractors that govern turbulent phenomena as we hoped to be a few years ago. The dream of building up a conceptual theory of the statistical properties of a steady turbulent flow as simple as statistical mechanics for conservative systems, without having to really understand the mathematically intractable fluid equations, appears for the time being quite remote, although some of the conference speakers have reported encouraging results and fresh new ideas on the subject.

Nevertheless, it seems to me that for a few more years the close interaction between theorists and experientalists has a good chance of continuing and of being fruitful. It appeared from several talks that theorists have still as their main objective understanding more deeply the properties of developed turbulence, and one should consider their preference for the onset phenomena as a preparation for more challenging problems. Much has in fact still to be understood in the low-degree-of-freedom systems that will continue to be relevant for the future theory of the statistical properties of strange attractors. Most of the talks were devoted to such technical problems as one would expect, since scientific progress can be only achieved "provando e riprovando" while waiting for the rare qualitative jumps in the understanding.

We are grateful to Accademia dei Linci for proposing the organization of the conference and providing the first cultural and financial support: Professor Luigi Radicati and the Centro Interdisciplinare Linceo have worked with particular interest for the birth of this meeting.

Then we thank IUPAP, IBM-ITAMIA, and the Mathematics Department of the II University of Rome for their prompt financial support. Finally, we are indebted to GNEM-CNR, to Comitato per la Matematica, and to the II University of Roma for further substantial financial help.

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Analysis of Vortex Methods for Incompressible Flow

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Vortex methods are used for the numerical simulation of time-dependent, incompressible flow without viscosity or at high Reynolds number. The principle is to solve a system of ordinary differential equations for the paths of representative particles in the fluid. This talk will survey the current state of theoretical work on these methods, emphasizing the formulation and error analysis for inviscid flow without boundaries. In this case the method can be designed so that the approximate solution converges with high-order accuracy for smooth flows in two or three space dimensions. A few test calculations will be described.

When boundaries are present, the effect of viscosity must be incorporated. A general method developed by Chorin[7] and Chorin and Marsden[8] uses elements of vorticity of two kinds which are advected according to a velocity computed from their current configuration. The interior flow is represented by a collection of vortices of finite core, or "blobs," while vortex sheets are generated near the boundary to satisfy the no-slip condition. A random walk of all the elements simulates the effect of viscosity. For a general descrition of this full method, see Ref. 8. For a survey of methods and applications, see Refs. 16 and 17. A concise summary of the formulation and theory for the inviscid case without boundary is given in Ref. 5. Related methods of "particle-in-cell" type have been in use for a long time in plasma physics as well as fluid mechanics; see Ref. 15.

Vortex methods are based on the fact that, for incompressible flow, the velocity field is the convolution of an integral kernel K wih the vorticity function ω. In two dimensions vorticity is conserved on particle paths. For this reason we can supose that a particle currently located at $x_j(t)$ carries a vorticity ω_j assigned at time zero. The vorticity is approximated at the current time by

$$\sum_j \varphi_\delta(x - x_j)\, \omega_j h^2$$

where h is the initial particle spacing and φ_δ is a smooth approximation to the δ function, scaled by another length parameter δ. This prescribed form of the vorticity leads to a system of ordinary differential equations

$$\frac{dx_i}{dt} = \sum_j K_\delta(x_i - x_j)\, \omega_j h^2$$

where $K_\delta = K * \varphi_\delta$. The velocity at arbitrary locations can be found in a similar way.

Methods of this type can be shown to converge with accuracy in good norms under suitable smoothness assumptions and with proper choices of parameters. The first results of this type were given by Hald and Del Prete[12,13] for two-dimensional flow. Beale and Majada proved convergence for one natural method in three dimensions and showed that the methods could be high-order accurate.[2,3] In three dimensions the vorticity must be updated as well as the locations of the elements.

A number of further improvements have been made in the theory. Cottet and Raviart[9,10,20] gave a much better treatment of consistency than the earlier ones. Cottet has also shown convergence of one version of the "cloud-in-cell" method. Anderson and Greengard[1] extended the convergence theory to include some methods of time discretization, and Hald has treated more general ones. Hald[14] has also shown convergence in two dimensions when the vorticity is only Holder continuous. Anderson and Greengard[1] discussed a second three-dimensional vortex method, and its convergence was shown in Ref. 6 and by a different method in Ref. 21. This latter method is less explicitly Lagrangian than that of Ref. 2. Greengard has shown convergence of another 3D method, closer to that of Ref. 2, in which the elements are introduced along vortex lines; he has used this version to simulate the interaction of two vortex rings Ref. 11.

In a different vein, Marchioro and Pulvirenti[18,19] have shown weak convergence of the random vortex method as an approximation to the equations of viscous flow. Recently, Goodman[22] has derived a stronger result of this type. In this case the particles satisfy a system of stochastic ordinary differential equations.

A simple class of test problem in two dimensions illustrates the strengths and weaknesses of these methods. The vorticity is chosen to be radial and varying so that there is shearing of the outside with respect to the inside. The predicted orders of accuracy are observed for moderate times. After a longer time, the accuracy deteriorates, but does not continue to get worse (Ref. 5). The distortion of the original configuration seems to be the largest source of error. There are a number of possible ways for improvement. One particularly simple way was discussed that is much more accurate and involves only a moderate increase in computation.

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Chaotic Behavior in One Dimension

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There is by now a good understanding of a rather general class of one-dimensional dynamical systems (iteration of maps of an interval). One can distinguish between the case of expanding mappings (modulus of the slope everywhere larger than one) which are very similar to uniformly hyperbolic systems, and the case of S unimodal mappings (which roughly speaking look like parabolas) corresponding to nonuniformly hyperbolic systems. We mainly review the properties of this class of transformations whose prototype is the logistic map $x \to 1 - \mu x^2$, $0 \leqslant \mu \leqslant 2$.

The topological approach is concerned with the nature of the attractors (periodic orbit, cantor set, intervals), while the ergodic approach investigates the statistical properties of trajectories. The most interesting questions are of course related to the so-called chaotic behavior. The first problem is the abundance of this chaos: In a one-parameter family of maps, how frequently do we meet chaotic behavior? The second problem concerns the statistical description of this chaos. Which invariant measure should we use? A satisfactory proposal was made some years ago by Bowen and Ruelle, and reemphasized recently by Milnor in connection with the topological approach. When the correct invariant measure is selected, what are the ergodic properties? Under some additional technical conditions we can now answer all these questions. Although many interesting problems remain open, the above picture should be a guideline for the analysis of higher- (but finite) dimensional dissipative systems.

We only quote a few recent articles and books. For the topological analysis of maps of an interval one can consult Refs. 13, 6, and 3. For more general systems, see Ref. 7. The statistical analysis is somewhat scattered in the literature. A review of the expanding case appeared in Ref. 4. The abundance question is treated in Refs. 8, 1, and 5. Invariant masures are discussed in Refs. 18, 6, 15, 11, 2, and 10. For recent results about higher-dimensional systems, consult Refs. 19, 7, 12, 17, 20, and 9. For infinite-dimensional systems, see Ruelle's contribution to this meeting.

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Oscillations in Solutions to Nonlinear Partial Differential Equations

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We discuss several aspects of a general program dealing with the structure of oscillations in solutions to nonlinear systems of partial differential equations. The topics deal with the analysis of solutions to quasilinear hyperbolic and elliptic systems of conservation laws arising in the theory of compressible fluid dynamics and hyperelasticity and in solutions to semilinear hyperbolic systems arising as discrete-velocity models in the kinetic theory of gases. The tools include the representing measure of L. C. Young and the Tartar-Marat theory of compensated compactness. We discuss various applications of the general theory dealing, for example, with convergnce of the viscosity method and convergence of classical finite difference schemes for hyperbolic systems of conservation laws in one space dimension.

We also discuss the notion of measure-valued solution to conservative systems of differential equations. The notion is relevant to the problem of analyzing various singular limits including the zero diffusion limit and zero

dispersion limit for systems of conservation laws. The prototypical models are provided by the singular transitions associated with Burger's equation and the *KdV* equation.

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1. L. Tartar, in R. J. Knops, ed., *Research Notes in Mathematics, Nonlinear Analysis and Mechanics*: Heriot–Watt Symposium, Vol. 4 (Pitman Press, 1979).

This paper provides an introduction to the Young measure and compensated compactness. It includes a proof of convergence of the viscosity method for a scalar conservation law using the aforementioned tools.

2. L. Tartar, in J. M. Ball, ed., *Systems of Nonlinear Partial Differential Equations*: NATO ASI Series (Reidel, New York, 1983).

This paper provides an introduction to the compensated compactness theory together with applications to convergence of the viscosity method for nondegenerated hyperbolic systems of conservation laws. Additional background information and applications to numerical methods can be found in the following paper.

3. R. J. DiPerna, *Arch. Rat. Mech. Anal.* **82**:27–70 (1983).

For results dealing with large-data existence and convergence of the viscosity for the isentropic equations of gas dynamics in one space dimension, we refer the reader to the following article.

4. R. J. DiPerna, *Comm. Math. Phys.* **91**:1–30 (1983).

For a general introduction to the weak topology in the setting of elasticity and the calculus of variations we refer the reader to the following two papers.

5. J. M. Ball, On the Calculus of Variations and Sequentially Weakly Continuous Maps, in *Lecture Notes in Mathematics*, Vol. 564 (Springer-Verlag, New York/Berlin, 1976).
6. J. M. Ball, *Arch. Rat. Mech. Anal.* **63**:337–403 (1977).

The concept of measure-valued solution and its relevance to the zero diffusion limit and the zero dispersion limit for conservation laws is discused in the following paper.

7. R. J. DiPerna, *Arch. Rat. Mech. Anal.* (1985).

As an introduction to a related question in compensated compactness and homogenization we refer the reader to the following introductory article, which deals with diffusion problems and discrete velocity models of the Boltzman equation.

8. L. Tartar, *Étude des Oscillations dans les Équation aux Dérivées Partielles Nonlinéares*, in Lecture Notes in Physics, Vol. 195 (Springer-Verlag, Berlin, 1983).

For basic structural results in compensated compactness we refer the reader to the following two articles.

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Chaos Through Quasi-Periodicity and Strange Attractors

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Nowadays, there is no longer any doubt about the existence of deterministic chaos; however, some questions remain, in particular, that one of the best ways of characterizing this chaos is from experimental data, together with the knowledge of the underlying physical mechanisms. From some examples, obtained from Rayleigh–Bénard convection in confined geometry, we try to address these questions in the case of the appearance of chaos from a biperiodic regime (from which it is known that different routes are possible).

The dynamics of two-coupled thermal oscillators near a phase locking (frequency ratio $\rho \simeq \frac{1}{7}$ and $\frac{1}{3}$) has been followed through the study of trajectories in the phase space. Phase intermittencies are well-evidenced, i.e., periods of nearly locked states, interrupted by fast rotations of the phase between the two oscillators. The time length of these periods may be constant (then no chaos is present) or stochastic, and then chaos appears. Phase intermittencies are also found in forced convection experiments where an artificial oscillator is induced and tuned near an intrinsic thermal oscillator. This phase dynamics is compared to properties of the numerical Curry–Yorke model (a mapping in R^2); the similarity between the experimental observations and the model is striking.

When the frequency ratio of the convective state departs from that of a locking state, a rich Fourier spectrum is measured near the onset of chaos. It may be understood in terms of continued fraction representation of the actual rotation number. In a very narrow range of Rayleigh number, Fourier spectra, Poincaré sections, and fractal dimension of the corresponding strange attractors have shown a drastic change of the dynamical behavior with the rotation number; the strong influence of the dynamical behavior with rotation number in the appearance of chaos is then clearly evidenced.

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References to Talk

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The existence problem for solutions of Feigenbaum's equation has been considered in Refs. 1 (the first detailed description of what is involved) 2, and 3, an analytic study which can be viewed as a precursor of the present work, and Ref. 4, where computer methods have been used for the first time.

The question of the limit as $n \to \infty$ of Feigenbaum's equation

$$\frac{1}{\lambda} \varphi_n \varphi_n(\lambda x) = \varphi_n(x),\ \varphi_n(x) = f_n(x'')$$

has been considered in Refs. 5–8.

The main ingredient of our analysis is an expansion $f_n(x) \sim 1 - h(z)/n$, leading to a fixed-point equation of the form

$$h[\Gamma(x)] - h(x) = J(x)$$

where $\Gamma(0) = 0$, $\Gamma'(0) = 1$.

This is the interesting case of a *Renormalization Group transformation with marginal eigenvalue and nontrivial fixed point.* We get a lot of information from Ecalle's work,[9,10] which we combine with Loeffel's work on Borel summability.[11,12] Our methods for computer-assisted proofs have been documented earlier[13]; see also Ref. 14 for a general discussion of interval arithmetic.

The above work is summarized in the joint publication with P. Wittwer.[15]

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Truncated Navier–Stokes Equations as Dynamical Systems: A Contribution to the Understanding of the Onset of Chaos

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In the past few years a new approach to the study of equations governing the flow of an incompressible fluid has gained ground thanks to the advent of modern computers. It is based on the numerical simulation of nonlinear systems of ordinary differential equations derived through truncation to a finite number of Fourier-components (modes) of suitable expansions of the partial differential equations that rule the fluid flow. Here we are concerned with the results that were obtained in studying suitable truncations of the planar Navier–Stokes equations with periodic boundary conditions. In particular, we briefly review some results which, in the framework of dynamical systems, seem to have given some contribution to the understanding of the onset of chaos.

The first significant result is found in a 5-mode truncation[1] and is associated to the discovery of sequences of presumably infinitely many period-doubling bifurcations of a periodic orbit. The process, which leads to the appearance of a strange attractor, is found to be compatible with the Feigenbaum theory of universality.

The phenomenon of period-doubling of a 2-torus is discovered in a 7-mode model.[2,3] Contrarily to what usually occurs in the case of a periodic orbit, the sequence of doublings stops after a finite number of steps. In the case we consider, a third bifurcation of a different kind generates an attractor which, shortly after, has a dimension larger than 3.

A detailed description of two "breakings" of a 2-torus is obtained from a 12-mode model.[4] The analysis of the intersection curve of the torus with a Poincaré section and the analysis of a return map associated with it, shows that the breaking is due to the appearance onto the curve of foldings, presumably in an infinite number, which make the structure of the attractor become quite complicated. The process is accompanied by frequency locking, which makes the definition of the transition point quite difficult.

In Ref. 4 some phenomena of "crisis" of a strange attractor are also described. Two of them are associated with the disappearance of the attractor arisen in consequence of the breaking of a torus. The crises, which are due to the collision with neighboring unstable periodic orbits, also in this case are evidenced by making use of a Poincaré section. In addition, one case is shown in which the crisis seems to involve a torus without being preceded by a breaking.

The asymptotic behavior of a sequence of resonances on a torus is investigated in Ref. 5. Correspondingly to different models (two 4-mode truncations and a 6-mode one) three sequences of lockings $n/(pn+1)$ show three distinct behaviors when, as n tends to infinity, they approach the final locking $1/p$. The behavior in fact goes as $1/n$, $1/n^2$, and c^n ($c \cong 0.7$), respectively.

Finally, we make a remark which comes from the phenomenologies described in Refs. 3 and 4. There can exist strange attractors on which the motion is the result of two components: a periodic component, which is due to two independent frequencies, and a chaotic one, which takes place on a small scale. In such a case, phase locking is very likely to occur and, furthermore, it may be quite difficult to detect chaos by the usual tools like spectral analysis. So, it may be practically impossible, particularly in experiments, to distinguish between quasiperiodic and such chaotic motions.

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Stability Near Resonances in Classical Hamiltonian Systems

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In this talk the results of a joint work with Benettin[1] are presented.

I consider a classical Hamiltonian system with l degrees of freedom. The Hamiltonian is supposed ε close to an integrable one.

The discussion will point out that the preceding work of Nekhorossev[2] not only sets up a bound on the time scale on which Arnold's diffusion can take place but also allows us to describe quite precisely many qualitative features of the motion up to a time scale $T_\infty(\varepsilon)$ depending on the perturbation strength ε so that

$$\varepsilon^k T_\infty(\varepsilon) \xrightarrow[\varepsilon \to 0]{} \infty, \qquad \text{for all} \quad k \geqslant 0$$

We define, following Nekhorossev, and then analyze the notion of "resonance of order r," $r = 0, 1, 2, \ldots, l$, as a set of data in phase space. We also analyze the behavior of the "slow" and "fast" variables associated with a resonance.

A precise meaning is given to the statement that within the time range $\varepsilon^{-1/2} \lesssim t \lesssim T_\infty(\varepsilon)$ the only nontrivial evolution is that of slow variables. Such evolution, when the initial data are reasonable, i.e., at least "double" (i.e., $r \geqslant 2$), can be described by a rather general Hamiltonian system with N degrees of freedom and generically not close to an integrable one. Chaotic phenomena are therefore possible after a time scale $\sim \varepsilon^{-1/2}$, and are controllable by perturbation theory.

If $N = 0, 1$, the time evolution stays very close to an integrable one up to the time scale $T_\infty(\varepsilon)$, and chaotic phenomena can show up only on larger time scales.

We recover all the results of Nekhorossev (in particular that Arnold's diffusion cannot start before the time scale $T_\infty(\varepsilon)$) with explicit values for all the constants under the simplifying assumption that the unperturbed Hamiltonian is either strictly convex or is linear nonresonant.

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The basic paper in modern perturbation theory for Hamiltonian systems is the paper of Kolmogorov[1]: in this work a rather general sketch of the proof of the first theorem of a series which is loosely called "KAM theory" is provided.

It seems that not all scientists agree on the completeness of such a proof: to help the reader understand the sketch of the proof by Kolmogorov, I suggest the paper by Benettin, Galgani, and Giorgilli[2] where the details are filled in.

The proof suggested by Kolmogorov is rather different from that given, for the same theorem, by Arnold[3] or Moser.[4]

Arnold's and Moser's new proofs are very interesting because they are based on a recursive method very close to what is today called a "renormalization group method." They were very important for the rather nontrivial extensions of the Kolmogorov theorem which led Arnold[5] to prove the possibility of a stable planetary system and Moser to prove the theorem under the sole assumption of high enough differentiability of the

Hamiltonian (as opposed to the analyticity assumed by Kolmogorov and Arnold).

A rather complete analysis of Arnold's method can be found also in Refs. 6 and 7 while a more precise connection with the renormalization group is worked out in Refs. 8 and 9.

It has been claimed that the KAM theory is useless from the point of view of the applications. I rather think that it has been too little understood by scientists interested in true applications. However, recently (i.e., starting in the late 1970s), many physicists have produced approximation schemes which, although usually nonrigorous from a mathematical point of view, follow rather closely the "KAM theory" and obtain bounds meaningful for the applications. I quote as examples Escande and Doveil,[10] McKay,[10] Siggia,[12] and Shender and Kadanoff.[13] Such works seem to indicate that the reasons why KAM theory has not been applied so far are of a technical nature and perhaps may be overcome.

Recently, a few workers have revisited the KAM theory with the purpose of obtaining better bounds both in the dependence on the number of degrees of freedom, see Wayne,[14] and on the actual value of the perturbation parameter ε when an invariant torus of given rotation number ceases to exist, see Porzio,[15] Liverani, Serviri, and Turchetti,[16] obtaining results which at least in the latter case are not too far from the best results obtained by other methods (i.e. by methods not based on proofs of the KAM theorem), see Aubry,[17] and Mather.[18]

In general, however, there still seems to be little contact between the mathematically rigorous works based on KAM theory and the ideas connected with the theoretical physics works mentioned above. I think that further research on this subject (namely on taking seriously the KAM theory as a source of useful bounds) should be done, in particular devoting attention to celestial mechanics problems where the rigorous bounds are particularly poor.

The theorem of Nekhorossev,[19] which has been the main subject of the talk, is a remarkable and not too well-known result complementary to the KAM theory, being the first clear quantitative formulation of the range of validity of perturbation theory as an asymptotic expansion in ε.

Our results[20] on the slow and fast variables are implicit in the Nekhorossev paper: technically we reverse the order of the proofs by treating first the harmonic oscillator case and then the general anisochronous systems. In neither case do we use the recursive technique of Nekhorossev. Our bounds are probably better than those of the Nekhorossev (not always very explicit and hence hard to use for the purpose of comparison). The l dependence in the case of anisochronous systems is the same found in Ref. 21, where the Nekhorossev theorem is proved with explicit evaluation of the constants along the lines of Nekhorossev work (i.e., using an iterative scheme).

The possibility of controlling the system up to an "infinite time scale" $T_{\infty}(\varepsilon)$ by perturbation theory might be useful in studying the effects of various models of dissipative forces on Hamiltonian systems and the related "phase lockings": in particular, one can examine friction models to see if the apparent abundance of systems observed in phase locking (i.e., very close to a resonance) can be explained by some principle of "minimum friction," as many have thought for some time. In this respect, see also the lecture by Ghil in this conference and the thesis by Wolansky.[22]

The possibility of extending the work presented in this talk to celestial mechanical cases has been discussed in the case of a restricted three-body problem by Celletti[23]: It would be interesting to extend this result to the full three-body problem.

Among the other basic open problems is that of finding techniques that enable us to deal with global "nonperturbative" phenomena like the Arnold diffusion: such a diffusion may just be outside the range of observability for small ε, it becomes important[24] when ε is large and the various time scales are no longer distinguishable. Of course, one thinks that the basic mechanism for Arnold's diffusion is the "wiskered tori,"[25,26] It is not clear, however, how a given system really contains such objects in the phase space.

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Mathematical Problems in Climate Dynamics

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Climate dynamics is a new and raidly developing geophysical discipline. Selected problems from this discipline are presented, starting with their physical motivation. Mathematical aspects emphasize nonlinear oscillations, entrainment and detrainment by periodic and quasi-periodic forcing, and the consequences of a complex frequency spectrum for predictability.

1. INTRODUCTION

Climate dynamics is a relatively new member of the family of geophysical sciences. Descriptive climatology goes back, of course, at least to the ancient Greeks, who realized the importance of the Sun's mean zenith angle in determining the climate of a given latitude belt, as well as that of land-sea distribution in determining the regional, zonally asymmetric characteristics of climate. The general human perception of climate change is also preserved in numerous written records throughout history, starting with the floods described in the epic of Gilgamesh and in the Bible.

Only recently did the possibility of global climate monitoring present itself to the geophysical community, through ground-based observational networks and space-borne instrumentation. This increase in quantitative, detailed knowledge of the Earth's current climate was accompanied by the development of elaborate geochemical and micropaleontological methods for sounding the planet's climatic past.

Observational information about present, spatial detail and about past, temporal detail were accompanied in the 1960s by an increase of computing power used in the processing of climatic data, as well as in the modeling and simulation of the seasonally varying general circulation of the atmosphere and ocean. The knowledge thus accumulated led to an increase of insight which was distilled in simple models, in an attempt to analyze the basic ingredients of climatic mechanisms and processes.

In the following lecture, we describe a few simple models and try to convey the flavor of the new, theoretical climate dynamics. As in every area of the exact sciences, the fundamental ideas suggested by simple models have to be tested by further observations and detailed simulations of the phenomena under study. We hope that this description of preliminary, theoretical results will stimulate the comparisons and verifications required to further develop the theory.

Theoretical climate dynamics as presented in this lecture are covered in Ref. 11 (Part IV, Chaps. 10–12), hereafter GC, and in Part V of Ref. 12. Specific references in the text are made to sections and figures in GC.

2. RADIATION BALANCE AND EQUILIBRIUM MODELS

The major characteristics of a physicochemical system, such as the climatic system, are given by its energy budget. The climatic system's energy budget is dominated by the short-wave radiation R_i coming in from the Sun, and the long-range radiation R_o escaping back into space. The approximate balance between R_i and R_o determines the mean temperature of the system. The distribution of radiative energy within the system, in height, latitude, and longitude, determines to a large extent the distribution of climatic variables, such as temperature, throughout the system.

We consider in this section a spatially zero-dimensional (0-D) model of radiation balance, with mean global temperature as the only variable. The dependence of the solar radiation's reflection on temperature, the so-called ice-albedo feedback, and the dependence of infrared absorption on temperature, the greenhouse effect, are discussed. Stationary solutions of this model and their linear and nonlinear stability are investigated (Refs. 4, 8, 16, 23, 29, and 30).

3. GLACIATION CYCLES: PHENOMENOLOGY AND SLOW PROCESSES

The previous section dealt with the climatic system's radiation balance, which led to the formulation of equilibrium models. Slow changes of these equilibria due to external forcing, internal fluctuations about an equilibrium, and transitions from one possible equilibrium to another have also been studied (Refs. 1, 22, and GC, Chap. 10).

Climatic records exist on various time scales, from instrumental records on the time scale of months to hundreds of years, through historical documents and archeologic evidence, to geological proxy records on the time scale of thousands to millions of years. These records indicate that climate varies on all time scales in an irregular fashion. It is difficult to imagine that a model's stable equilibrium, whether slowly shifting or randomly perturbed, can explain all this variability.

A summary of climatic variability on all time scales appears in Ref. 19. The most striking feature is the presence of sharp peaks superimposed on a continuous background. The relative power in the peaks is poorly known; it depends of course on the climatic variable whose power spectrum is plotted, which is left undefined in GC, Fig. 11.1. Furthermore, phenomena of small spatial extent will contribute mostly to the high-frequency end of the spectrum, while large spatial scales play an increasing role toward the low-frequency end.

Many phenomena are believed to contribute to changes in climate. Anomalies in atmospheric flow patterns affect climate on the time scale of months and seasons. On the time scale of tens of millions of years, plate tectonics and continental drift play an important role. Variations in the chemical composition of the atmosphere and oceans are essential on the time scale of billions of years, and significant on time scales as short as decades.

The appropriate definition of the climatic system itself depends on the phenomena one is interested in, which determines the components of the system active on the corresponding time scale. No single model could encompass all temporal and spatial scales, include all the components, mechanisms, and processes, and thus explain all the climatic phenomena at once.

Our goal in this lecture is much more limited. We concentrate on the most striking phenomena to occur during the last two million years of the Earth's climatic history, the Quaternary period, namely on glaciation cycles. The time scale of these phenomena ranges from thousands to millions of years. We attempt to describe and model in the simplest way possible the components of the climatic system active on these time scales —atmosphere, ocean, continental ice sheets, the Earth's upper strata—and their nonlinear interactions.

We sketch the discovery of geological evidence for past glaciations, review geochemical methods for the study of deep-sea cores, and describe the phenomenology of glaciation cycles as deduced from these cores (Ref. 7, GC, Chap. 11, Ref. 17). A near-periodicity of roughly 100,000 years dominates continuous records of isotope proxy data for ice volume, with smaller spectral peaks near 40,000 and 20,000 years, as suggested in GC, Fig. 11.1. The records themselves are rather irregular and much of the spectral power resides in a continuous background (GC, Sec. 11.1; Ref. 15).

Next, we give a brief introduction to the dynamics of valley glaciers and large ice masses. The rheology of ice is reviewed (Glen, 1955) and used in deriving the approximate geometry of ice sheets. The slow evolution due to small changes in mass balance of an ice sheet with constant profile is modeled next.[3,24,25] A simplified, but temperature-dependent formulation of the hydrologic cycle and of its effect on the ice mass balance is given. We study multiple equilibria of the ice-sheet model thus formulated and their stability pointing out similarities with the study of energy-balance models in Section 2. A hysteresis phenomenon occurs in the transition from one equilibrium solution to another as temperature changes (Ref. 9, GC, Sect. 11.2).

Finally, we study the deformation of the earth's upper strata under the changing load of ice sheets. The rheology of lithosphere and mantle is reviewed. Postglacial uplift data and their implications for this rheology are outlined. A simple model of creep flow in the mantle is used to derive an equation for maximum bedrock deflection under an ice sheet and for the way this deflection affects the mass balance of the sheet (GC, Sect. 11.3, Ref. 26, Ref. 31).

The equations derived and analyzed in this section for ice flow and bedrock response will lead, when combined with an equation for radiation balance from the previous section, to a system of differential equations which govern stable, self-sustained, periodic oscillations. Changes in the orbital parameters of the earth on the Quaternaly time scales provide small changes in insolation (Ref. 2, GC, Sect. 12.3). These quasiperiodic changes in the system's forcing will produce forced oscillations of a quasiperiodic or aperiodic character, to be studied in the next section. The power spectra of these oscillations show the above-mentioned peaks with periodicities near 100,000, 40,000, and 20,000 years, as well as the continuous background apparent in the data.

4. CLIMATIC OSCILLATIONS

In Section 3 we reviewed some of the geological evidence for glaciation cycles during the Quaternary period. Large changes in global ice volume and changes of a few degrees in global mean temperature have occurred repeatedly over the last two million years. It is these changes we would like to investigate in the present section, with the help of very simple models.

These simple models do not represent the definitive formulation of a theory for climatic variability on the time scales of interest. They are used merely to illustrate some ideas we believe to be basic for an understanding of this variability, an understanding which is still in the early stages of development. Other models and related ideas can be found in the references of Section 5.

We formulate and analyze a coupled model of two ordinary differential equations for global temperature and global ice volume. The equations govern radiation balance (Sect. 2) and ice-sheet flow (Sect. 3), respectively. This model exhibits self-sustained oscillations with an amplitude comparable to that indicated by the records and a period of 0(10 ka), where 1 ka = 1000 years. Phase relations between temperature and ice volume and their role in the oscillation's physical mechanism, are investigated (Ref. 10 and GC, Sect. 12.1). Stochastic perturbations of such self-sustained climatic oscillations have also been considered.[22,28]

Exchange of stability between equilibria (Sect. 2) and limit cycles (Sect. 3) in models with an arbitrary number of dependent variables and spatial dimensions is studied next. The distinction is made between a stable limit cycle which grows slowly in amplitude from zero as a parameter is changed (direct or supercritical Hopf bifurcation) and sudden jumps from zero to finite amplitude (reverse of subcritical Hopf bifurcation). Structural stability and the special role of limiting, structurally unstable, homoclinic, and heteroclinic orbits is discussed (GC, Sect. 12.2).

We introduce the geometry and kinematics of orbital changes in the earth's motion around the sun from the perspective of the small insolation changes they generate. Eccentricity, obliquity, and precession of the earth's orbit are defined. A few fundamental concepts of celestial dynamics and the associated mathematical methods are reviewed (GC, Sect. 12.3).[5,32] We report currently accepted results on the periodicities of insolation changes during the Quaternary period: 19 and 23 ka for precession, 41 ka for obliquity, and 100 and 400 ka for eccentricity. Their action on the climatic system's radiation balance and hydrologic cycle is modeled.

Finally we take up the effects of this action upon the climatic oscillator above, augmented by a third equation, governing bedrock response to ice load (Sect. 3). The free oscillations of this model are found to differ but little from those of the previous one. Eccentricity forcing is shown to produce a very small or very large response according to whether the system operates in an equilibrium or in an oscillatory mode (GC, Sect. 12.4).

We study in detail the internal mechanisms by which forcing at one or more frequencies can be transferred through the system to additional frequencies, as well as to the climatic spectra's continuous background. Entrainment results in the system's free frequency becoming locked onto an integer or rational multiple of a forcing frequency. Loss of entrainment leads to aperiodic changes in system response.

Combination tones are linear combinations with integer coefficients of the forcing frequencies. Prominent among them in the data are those with periods near 15, 13, and 10 ka, and their harmonics.[27] As a result of hydrologic and insolation forcing at the orbital frequencies, the model considered here produces spectral lines at many of these observed frequencies, superimposed on a continuous background associated with aperiodic, irregular terminations of glaciated episodes (Ref. 10 and GC, Sect. 12.5).

Finally we consider the predictability and reproducibility of climatic time series. The consequences of multiple spectral lines and of the continuous background for predictability, or the lack thereof, are investigated. It is argued that irretrievable loss of predictive skill over a time interval 0(100 ka) is intrinsic in the aperiodic nature of Quaternary climate changes (GC, Sect. 12.6).

Related ideas for climate dynamics on shorter time scales (months to years) are presented in this volume by Legas in GC, Chaps. 5 and 6, Ref. 12, Part IV, and Refs. 6 and 18. The use of dynamical systems with discrete-valued variables in modeling certain qualitative aspects of climate dynamics is summarized also in this volume.[20]

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Symmetry, Entropy, and Coherence in Chains of Oscillators

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Under very general assumptions, long chains of weakly coupled limit cycle oscillators can phase-lock if the frequency difference among the oscillators are not too great. The amount of frequency variation that can exist without eliminating locking depends on the symmetry properties of the oscillators; in particular, the scaling behavior (as the number of oscillators increases without bound) depends on the symmetry. The analysis uses an unusual continuum limit of the discrete equations for the phase differences of the oscillators. The proof that this continuum limit is a correct diagnostic equation points up parallels with numerical problems concerning algorithms for computing the correct ("entropy") weak solutions to scalar quasilinear P.D.E.s.

The equations examined have the form

$$\dot{X}_k = F_k(X_k) + \varepsilon[G^+(X_{k+1}, X_k) + G^-(X_{k-1}, X_k)] \tag{A1}$$

where $X_k \in R^m$; $G^{\pm}$, $F_k\colon R^m \to R^m$ are smooth, $\varepsilon \ll 1$, and $k = 1,..., N+1$. The chain is finite, so

$$G^-(X_0, X_1) \equiv 0, \qquad G^+(X_{N+2}, X_{N+1}) \equiv 0 \tag{A2}$$

$\varepsilon \ll 1$ implies that (A1) can be reduced to equations of the form (2) in the Bibliography; see Section 2 of the latter. More details about the analysis of phase-locked solutions of the above equation are also found in Section 2.

Different kinds of coupling lead to qualitatively different solutions. In particular, for biological applications there is an important distinction between "diffusive coupling" $[G^{\pm}(Y, Y) \equiv 0]$ and synaptic coupling $[G^{\pm}(Y, Y) \not\equiv 0]$. Many detailed observations about fish locomotion, which is thought to be governed by such a chain of neutral oscillators, follow from equations as general as (A1), provided the coupling is synaptic.

Phase Methods for Coupled Oscillators and Related Topics: An Annotated Bibliography

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Most of the work on oscillators to be commented on here concerns limit cycle oscillators, i.e.; dissipative, not Hamiltonian, systems. The dissipation is not small; rather, the oscillators have quite stable periodic solutions, but the coupling among them is weak. In this limit, much can be said about the behavior of the coupled system with almost no hypotheses on the oscillators.

1. PHASE METHODS: TWO OSCILLATORS

Weak coupling allows the equations for the full coupled system to be replaced by a much lower dimensional system in which each oscillator is represented by only its phase. For a forced oscillator, this was done in 1950 by Levinson.[1] Independent of the differential equations for the forced oscillators, the phase space reduces to a two-dimensional torus, with equations having the form

$$\begin{aligned} \dot{\theta}_1 &= \omega_1 + \varepsilon H(\theta_0 - \theta_1) + O(\varepsilon^2) \\ \dot{\theta}_0 &= \omega_0 \end{aligned} \tag{1}$$

Here θ_1 and θ_0 are the phases of the forced and forcing oscillators, $\omega_1 - \omega_0 = O(\varepsilon)$, $\varepsilon \ll 1$, and H is a 2π-periodic function explicitly computable from the original equations. Similar equations can be derived for a pair of weakly coupled oscillators, using averaging methods[2] or formal techniques involving "suppression of secular terms."[3,4]

Using such equations, it is possible to find conditions for "phase-locking," i.e., solutions in which both oscillators move at frequencies that are rationally related, with phase differences that are stably periodic in time.[5,6] The limits of phase-locking depend not only on the limit cycle, but also on the trajectories of the forced oscillator near the limit cycle.[7] When locking fails, there can be drift or "phase walk-through."[3,4,8]

For a forced oscillator or a pair of coupled oscillators, the relevant phase space is a two-dimensional torus, and the Poincaré map[9] is a circle map. Phase-locking corresponds to stable periodic points of the Poincaré map, so information about the existence of locking can be obtained from the theory of circle maps.[10–13] Poincaré maps do not give all possible information about the coupled systems. For example, if two oscillators of different frequencies phase-lock, one would like to know at what frequency the coupled system runs. In the weakly coupled limit, this frequency can be computed and need not lie in the interval between the two natural frequencies.[14]

Equations of the form (1) (or similar equations for two mutually coupled oscillators) fail to describe the full equations if the strength of the limit cycle is comparable to the coupling strength (as in strongly forced or weakly nonlinear oscillators[15,16] or if there is a very fast time scale in the forcing[17] or in the oscillators themselves (as in relaxation oscillators.[18–21] Nevertheless, phase methods can still be used. In some cases, there is still a Poincaré map, but it is no longer invertible[17,22] and displays some of the chaotic phenomena associated with one-dimensional maps.[23,24]

2. LARGER COLLECTIONS OF OSCILLATORS

Using results from invariant manifold theory,[25] it can be shown that if limit cycle oscillators are weakly coupled $[O(\varepsilon)]$, then the phase space has a stable invariant torus whose dimension is the number of oscillators $(N+1)$. (The size of the allowable ε is uniform in N if the coupling among the oscillators is local, e.g., nearest neighbors, rather than in a mean field theory[26]). If the oscillators are arranged in a chain, the dynamics of the oscillators are uniformly close, and the coupling is nearest-neighbor; then, to lowest order in ε, there is a set of equations similar to (1) for the N-phase differences.[27] These equations have the form

$$d\phi_k/d\tau = \Delta_k + H(\phi_{k+1}) - H(\phi_k) + H(-\phi_k) - H(-\phi_{k-1}) \tag{2}$$

$$H(-\phi_0) = 0 = H(\phi_{N+1}) \tag{3}$$

Here $\varepsilon\Delta_k = \omega_{k+1} - \omega_k$, where ω_k is the frequency of the kth oscillator; H is a 2π-periodic scalar function which can be computed using averaging techniques.[2] $\phi_k \equiv \theta_{k+1} - \theta_k$, where θ_k is the phase of the kth oscillator.

For such chains of oscillators, the behavior of the system depends on the symmetry properties of the function H. If H is an odd function of ϕ, phase-locked solutions to (2) are easy to find when they exist (Refs. 27 and 28). (By "phase-locked" we mean here "1–1 phase-locked," i.e., all the oscillators run at the same frequency, with phase differences independent of time). If, in addition, $\Delta_k \equiv \Delta$ (a linear frequency gradient), then for any fixed Δ and N large enough, there are no phase-locked solutions; the total frequency difference from one end of the chain to the other that can be sustained without losing locking is $O(1/N)$. If H also has an even component, this changes dramatically.[26,29] In this case, the phase-locked solutions are no longer easy to find, and will be discussed further in Section 3. For some examples of a physical interpretation of the symmetry or lack of it in H, see Ref. 29.

If $\{\Delta_k\}$ is too large to allow phase-locking, the solutions to (2) may still maintain local coherence. In Ref. 27, eqs. (2) are investigated for $H = \sin\phi$, $\Delta = \Delta(N)$ just large enough to prevent locking. It is shown that there are "frequency plateaus," or stretches of oscillators, for which the frequency is constant, with a break at which the frequency changes. This is the analogue, for N oscillators, of the "drift" between two oscillators of sufficiently different frequencies.[3,4] Such plateaus are seen numerically for equations involving coupled van der Pol oscillators in the sinusoidal regime.[30] See Ref. 27 for related references; these studies were all motivated by observed frequency plateaus in peristaltic movement.[31] Another paper which examines phase-locking in chains of coupled oscillators is Ref. 32.

Phase-locking has also been investigated for large collections of oscillators coupled more globally, e.g., to all other oscillators. Neu (Ref. 33) used phase methods, plus integrodifferential equations, to study how identical oscillators with varied initial conditions reach synchrony. Ermentrout,[34] studied oscillators with random frequencies, coupled sinusoidally ($H = \sin\phi$), and found limits on the variation of the natural frequencies in order for synchronization to occur. Winfree (Refs. 35 and 36, Chaps. 4 and 8, respectively) has done many computer simulations of collections of coupled oscillators and seen that, for some kinds of oscillators and coupling, synchronization does not occur, and the oscillator phases are widely dispersed. This might be expected for, e.g., a population of identical oscillators with coupling by a coupling function H for which $H'(0) < 0$, since the synchronous solution is unstable. (This can be seen from the linearization of the mean-field analogue of Ref. 2). Traub[37] has examined collections of oscillators randomly coupled to understand the synchronization of neurons that occurs in epilepsy. (See also Refs. 38 and 39.)

Finally, we mention work on coupled circle maps and coupled logistic maps.[40,41] We note that a system of two coupled oscillators described by O.D.E.s is reducible, via a Poincaré map, to a single circle map. However, larger collections of such oscillators do not (by standard means) reduce to lower-dimensional collections of coupled maps.

3. PHONY CONTINUUM EQUATIONS AND ENTROPY CONDITIONS

The phase-locked solutions to the equations for a chain of weakly coupled oscillators correspond to the critical points of (2). Unless H is an odd function of ϕ there is no explicit way to compute these critical points, or even to determine, for given H and frequency differences Δ_k, whether there are any. However, if the number of oscillators is large, it can be rigorously shown,[26] under a certain hypothesis that there *is* an asymptotically stable time-independent solution to eqs. (2) and (3) provided there is an asymptotically stable time-independent solution to a continuum problem of the form

$$\phi_\tau = \beta(x) + 2f(\phi)_x + (1/N)\, g(\phi)_{xx} \tag{4}$$

$$f = g \quad \text{at } x = 0; \qquad f = -g \quad \text{at } x = 1 \tag{5}$$

Here $0 \leqslant x \leqslant 1$, $x \approx k/(N+1)$, $\phi_k \approx \phi[k/(N+1)]$, and $N\Delta_k \equiv \beta_k \approx \beta[k/(N+1)]$. If $H^+ = H^-$ as in (2), then f and g are the even and odd parts of H (so $H = f + g$); if $G^+ \neq H^-$, the functions f and g are computable from H^+, H^- in a somewhat more complicated way. The time-independent solutions to the discrete problem eqs. (2), (3) converge as $N \to \infty$ to the corresponding solution to eqs. (4) and (5).

When N is large, the time-independent version of eqs. (4) and (5) constitutes a singularly perturbed two-point boundary value problem[42] that can be solved uniquely (for a stable solution to eq. (4)), under a very general hypotheses, provided that $\int_0^1 \beta(x)\, dx$ is not too large.[26] In general, there is a shock layer whose size is $O(1/N)$. The extra hypothesis mentioned above, needed to ensure that the solutions to the discrete problem approximates that of the continuous one, is that $g' > |f'|$ along the relevant solution to (4), (5). This can be thought of as a stability criterion for algorithms that discretize (4); it says that the shock layer of (4), whose size is governed by $g'(\phi)/Nf'(\phi)$ near the layer, is "large enough" relative to the mesh size $1/N$. It is also closely related to entropy conditions for numerical algorithms for solving $u_t = F(u)_x$ that ensure convergence (as the mesh size decreases) to the right weak solutions.[43,44] Osher[45] has written on discretizations of singularity perturbed boundary value problems; his work, aimed at solving the continuum problem, uses one-sided discretization schemes. It can be shown that the O.D.E. (2) can be written as a central differencing scheme for (4), to which Ref. 45 does not apply.

Under the conditions $g'(\phi) > |f'(\phi)|$ eq. (2) turns out to be a monotone scheme. Maximal principle ideas are then relevant. (See Ref. 46 for use of the maximal principle in connection with singularity perturbed problems.) These can be adapted to show the local asymptotic stability of the solutions to (2), (3); they are not easily used to prove the *existence* of time-independent solutions because it is unclear how to construct upper and lower solutions until real solutions have been constructed.

4. REAL CONTINUUM EQUATIONS

The continuum limit (4), (5) does not correspond to a physical continuum; it is only a "diagnostic equation" for (2), (3). However, the mathematics is related to mathematics describing real continua, in particular, reaction-diffusion equations[47]

$$\mathbf{u}_t = \bar{f}(\mathbf{u}, \mathbf{x}) + D\Delta^2\mathbf{u}, \qquad \mathbf{u} \in R^n, \qquad \mathbf{x} \in R^k \tag{6}$$

If the space coordinate is one-dimensional, and the reaction dynamics

$$\mathbf{u}_t = \bar{f}(\mathbf{u}, x) \tag{7}$$

has a stable limit cycle solution for each x, then the standard discretization of (6) is a chain of oscillators coupled to its nearest neighbors. As the mesh size decreases, however, the coupling between adjacent elements grows in strength without bound. This is not true of the equations described earlier, in which adjacent oscillators remain weakly coupled for all N; thus the scaling of the equation as $N \to \infty$ is different.

If the frequency variation (as x is changed) is not too great, trajectories can be expected to say, for each $\mathbf{x}$, near the limit cycle of (7). In this case, phase methods are useful for describing the interactions of the complicated wave patterns that form.[42–50] For a class of simple kinetic equations, this can be done rigorously[51–53]; for a much larger class of equations, it can be done formally through reduction of the equations to phase equations related to Burger's equation.[53–56]

Burger's equation predicts phase-locking no matter how large the frequency spread; however, for a large frequency difference, the formal derivation can no longer be expected to be valid and phase-locking does, indeed, break down, by mechanisms that involve changes of amplitude. Similarly, for patterns such as spirals[48,57–59] in which the phase is not well-defined at each point, phase methods are helpful but not completely sufficient. For some reaction equations having relaxation oscillations, an elegant analysis of spiral waves has been by Keener (to appear) using an Eikonal equation for the wave front.

5. BIOLOGICAL PHENOMENA

Many of the papers mentioned above were motivated by or related to biological problems involving oscillators. (For a large bibliography, as of 1980, see Ref. 36.) Equations describing propagation of action potentials down a nerve axon[60,61] are closely related to reaction-diffusion equations. Problems involving cardiac dynamics involve both forced systems[17,22] and sheets of electrically excitable tissue; the recent work of Keener on spiral waves is relevant to the latter. Large collections of oscillators with phases spatially distributed are important in systems of smooth muscle, including the intestine,[30,31] stomach, ureter, and small arteries. Such collections of oscillators are also implicated in Central Pattern Generators (CPGs), the autonomous programs in the central nervous system which govern sterotypic rhythmic motions such as walking, running, chewing, and breathing.[62–63] Much of the work described in the abstract was motivated by these two classes of systems, in particular by peristalitic motions in the intestine and fish locomotion.[64] For further mathematical problems and references involving CPGs, see Ref. 14.

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Renormalization Group Methods for Circle Mappings

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This talk surveyed the application of renormalization group methods to the analysis of iteration of smooth circle mappings with critical points, starting with special rotation numbers like the golden ratio and extending to general rotation numbers. A detailed text for a closely related talk will appear in the proceedings of the conference: Statistical Mechanics and Field Theory: Mathematical Aspects, held in Groningen in August of 1985, to be published in Springer Lecture Notes in Physics.

Microscopic Dynamics and Macroscopic Evolutions

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The derivation of laws describing the observed behavior of macroscopic systems from the fundamental ("true") laws governing the dynamics of electrons, nuclei, atoms, etc., multitudes of which make up every piece of macroscopic matter, is the central objective (holy grail) of statistical mechanics. This goal has been achieved, to a remarkable degree, for equilibrium systems. Macroscopic properties are identified with the "almost sure" values of appropriate observables (functions of the microscopic state of the system) with respect to known probability measures (functions of the system's Hamiltonian). This requires, of course, some kind of thermodynamic, infinite volume limit in which the distinction between microscopic and macroscopic becomes precise.

The situation is far less satisfactory at present for the more complex phenomena encountered in nonequilibrium systems, e.g., those described by the Boltzmann and/or the Navier–Stokes equations. While even the conceptual problems involved in such a derivation (time-reversible microscopic laws leading to time-irreversible macroscopic equations) are not entirely resolved the principal problems are (I believe) technical. Useful progress can therefore be made by considering model systems with stochastic microscopic dynamics, e.g., particles on a lattice evolving according to some Glauber and/or Kawasaki dynamics or interacting Ornstein–Uhlenbeck particles in the continuum. Some of these models have the additional interest of approximately describing other phenomena, e.g., biological or social, in which the individual entities are themselves macroscopic.

The current status of this program is the main topic of this session. I also discuss what these systems can teach us about the behavior of real systems, both microscopic and macroscopic.

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Large-Scale Atmospheric Dynamics

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We present a brief review on some selected topics of the dynamics of large-scale atmospheric flows. A general introduction to the field may be found in Pedlosky,[48] Gill,[18] and Hoskins and Pearce.[25]

PREDICTABILITY

As any chaotic flow, the atmospheric circulation possesses a sensivity to the initial conditions which limit in practice our ability to predict its future evolution. This concept is central in weather forecasting and has appeared very early in meteorology. Thompson[59] was the first to discuss the impact of the error-doubling time of the forecasts on the design of the operational network. Lorenz,[39] in a pioneering work, showed that a simple deterministic system of three ordinary differential equations was sufficient to produce an unpredictable behavior.

The amplification of noise by the dynamics was further studied from the statistical standpoint.[35,36,40] The initial error concentrated with the small scales was shown to propagate and to invade progressively the whole spectrum. The growth rate was found exponential in two-dimensional turbulence but leads to a finite time of predictability in the three-dimensional case. This latter result was put in relation with the conjectured lost of analycity of Navier–Stokes equations at a finite time in the limit of large Reynolds number.[53] More recently, Frisch[16] discussed the intermittency and the singularities of Navier–Stokes equations in terms of singularities in the complex space-time domain.

Extensions of the statistical approach were performed by several authors to investigate the effect of wave propagation[3,22] or stratification,[63] The predictability of large-scale structures in decaying turbulence was discussed by Metais, Chollet, and Lesieur.[47]

In parallel, direct evaluations of the atmospheric predicability from general circulation models were obtained by Smagorinsky[56] and Lorenz.[42] All estimates indicate that the upper limit of significant weather forecasting using the present observational network is between 10 and 15 days.

However, the quality of the forecast depends strongly on the season and the geographical location.[19] It also depends on the average flow configuration.[65] In simple models,[33] the deterministic predictability varies from one art of the attractor to the other, in relation with the flow regime. The predictability of the forecast errors is a challenge for operational weather centers. The prediction of the error covariance is in principle a by-product of the Kalman filters discussed for data assimilation by Cohn, Ghil, and Isaacson.[8] Stochastic-dynamic methods[14,60] or Monte-Carlo methods[21] have also been proposed together with purely deterministic variational approaches.[32,38]

Finally, extended range predictability beyond the deterministic limit appears possible on a statistical basis for transitions between weather regimes (see Sect. 3), or when the boundary conditions—sea surface temperature, soil moisture, snow cover,...—are varying over a long time scale and induce climatic drifts.

Numerous other references on atmospheric predictability are collected in two volumes.[17,23]

INVARIANT SLOW MANIFOLDS

The eigenmodes of the atmospheric flow split into two families, the gravity modes and the Rossby modes. These latter are due to the variation of the Coriolis force with latitude. Within the scales of most common meteorological perturbations, around 1000 km, the associated time scales separate clearly: a few hours for the gravity modes and several days for the Rossby modes. It results from observations that the atmospheric data, for instance the surface pressure, do not show any significant oscillations peaking within the range of gravity frequencies. This is also true for the regime state of a realistic general circulation model. However, when raw observations are used to initialize this GCM, large amplitude gravity oscillations generally develop as transients which persist over a few days.

It is thus widely admitted that in operational forecasting models the filtering of gravity waves must be a part of the data assimilation process. The first investigtors have soon recognized that it is not sufficient to project the initial data onto the subspace of Rossby modes to get rid of the transient oscillations. Indeed, they develop rapidly in this case through nonlinear interactions. In the nonlinear normal mode initialization,[43] the difficulty is cured through an approximate cancellation of the first time derivatives of the fast modes. This algorithm, presently implanted in most operational numerical models, leads to a significant improvement over the first days of the forecast.[58] However, no superiority is observed with respect to a noninitialized model after four days.

The generalization to higher-order derivatives was first performed by Baer and Tribbia[2] and subsequently formalized by Leith[37] and Lorenz.[41] They introduced the idea that the dynamics of the large-scale real atmosphere is entirely contained in a manifold of its phase space which has the dimension of the Rossby modes subspace. On this manifold, the gravity modes are slaved to the Rossby modes, so that all observed quantities vary slowly with time. Starting from any initial state, its projection onto the slow manifold is computed through an iterative expansion obtained by successive approximate cancellations of the *n*th order time derivatives of the linear gravity modes. When applied to simplified atmospheric models (Tribbia, 1979; Ballish, 1981), the second-order algorithm leads to a significant improvement of the filtering of transient gravity modes with respect to Machenhauer's method. Lorenz[41] applied a high-order algorithm to a nine-components primitive equation model and was able to reconstruct the observed attractor up to a precision of five digits.

There is obviously some mathematical interest in the possible existence of an invariant attracting manifold in the atmospheric equations. There is also a need for clarification since very few rigorous results have been established so far. We raise here a few of questions which may be addressed within this topic.

Within small scales, gravity wave activity, in particular associated with orography and convection, cannot be neglected in the atmosphere although it mainly escapes the standard observational network. This is especially true at tropical latitudes. There is clear evidence that this activity is not completely slaved to the large-scale circulation but depends also on small-scale fluctuations. Thus invariant manifolds may only exist for a set of atmospheric equations where these processes are parameterized and not explicitly represented.

The existence of an attracting invariant manifold is only proved in some cases for simplified atmospheric models.[29] There are numerical evidences[13,64] that gravity waves are commonly generated by intermittent transfers from the Rossby modes. Their apparent nature, slave or free, is strongly dependent on the damping factor.

The concept of slowness is not well-defined. Generally, the high-order time derivatives of a function are not algebraically bounded. Then the series used in initialization are asymptotic and not convergent. When local invariant manifolds exist, analytic expansions are possible.[64] However, these expansions become also asymptotic when the manifolds are not analytic.

MULTIPLE REGIMES

A noticeable part of the atmospheric variability of mid-latitude lies in the time scale domain from 10 days to a few months,[15] well beyond the average life span of cyclones of three to five days. This variability is mainly quasi-stationary and concentrated at a few preferred geographical locations.[5] Its statistical distribution of life span is exponential in character.[10]

Among persistent anomalies, the blocking pattern, as documented by Rex,[52] consists in the appearance of a quai-stationary center of high pressure, located at about 50°N in certain preferred areas, off the western margins of the continents. This blocking high may persist for longer than 10 days. It deflects the traveling cyclones from the usual storm tracks and produces a strong southward advection of polar air on its eastern flank, inducing severe cold episodes in winter. The occurrence or the non-occurrence of this feature determines to a large extent the distinctive character of an individual season; it is therefore of great importance to weekly and monthly mean weather prediction.

Egger[11] was the first to suggest that the internal atmospheric variability, modeled by nonlinear wave–wave interactions, could account for the finite amplitude and duration of blocking events. Charney and DeVore[6] took the more general view that blocking and near zonal flow could be associated with two distinct stable stationary solutions of a highly simplified atmospheric model.

Although being still controversial, the existence of multiple regimes in mid-latitude circulation is substantiated by the observation of bimodality in the atmospheric data.[4,10,20] Here the main difficulties lie in the absence of sharp separation between regimes and the shortness of available data series.

Multiple regimes have so far been mainly investigated in simple but progressively refined models. In one-layer models, the main ingredient is the coupling between waves and mean flow by an orographic form-drag. Introducing sphericity[28] and more degrees of freedom,[33] quite realistic patterns obtain. The impact of the truncation on the bifurcation diagram is discussed by Yoden.[66] The deformation of linear resonances is discussed by Malguzzi and Speranza[45] and by Benzi et al.[4] Pierrehumbert and Malguzzi[50] show that the existence of finite amplitude local solutions of the Euler equations, like the Batchelor's dipole, may explain multiple flow regimes without need of resonance. Local stratified solutions are also obtained by Malguzzi and Manalotte–Rizzoli[46] as eigenmodes of a Schrödinger equation. Tung and Rosenthal[62] discuss the physical range of validity for barotropic models. Indeed, correct energetics obtains only when one consider stratified models.[7,51] In particular, Itoh[27] shows that realistic exchanges are only observed after the first chaotic transition, and Legras and Vautard[54] extend the approach to the maintenance of large-scale stationary regimes by small-scale free transients.

In these studies, multiple flow regimes are obtained as stationary stable solutions or as pieces of more complex attractors. The relevant characteristic is then the concentration of measure in several subdomains of the phase space, each of which corresponding to a separate large-scale pattern. Unstable stationary solutions are often landmarks of persistent flow regimes in the attractor. No detailed knowledge of the attractor is needed except possibly in the connecting area between two pieces. Transitions between regimes or breaks are induced preferentially by small-scale activity,[12,51] their statistical distribution bearing similarities with a Markov process.[33] The exit time from a given flow regime characterizes its regime predictability which must be distinguished from the more familiar concept of pointwise-predictability (Sect. 1). This latter depends on the detailed dynamics and may be different from one flow regime to the other.

MAINTENANCE OF LARGE-SCALE STRUCTURES BY SMALL-SCALE TRANSIENTS

The idea that large-scale structures may be maintained by small-scale transients dates back to the concept of negative eddy viscosity discussed by Starr.[57] In the classical phenomenology of two-dimensional turbulence,[30] the energy is cascaded backward to the large scales when the enstrophy (the square of the vorticity) is cascaded to the small scale and dissipated. Numerical simulation[3,44] and experimental results[9] show that long-live localized eddies tend to be produced by this process. Inside these eddies the vorticity is strongly correlated with the streamfunction, inhibiting the nonlinear interactions.

In the atmosphere, the transients act on the average to dissipate the stationary waves.[31] However, observations indicate that they play a crucial role in the maintenance of the blocking pattern.[25,26] Theoretical studies show that at least three mechanisms are candidates for this effect.

1. The straining of barotropic disturbances propagating on a diffluent jet may enhance local nonlinear transfers and reinforce the splitting of the basic flow.[54]
2. At mature development stage, baroclinic disturbances are able to induce southward potential vorticity fluxes.[55] This may account for the location of blocking highs at the end of the storm tracks and the multiplicity of the regimes owing to the existence of a feedback loop between large-scale circulation and small-scale stresses.[34]
3. Similar effect is obtained in a pure linear framework when local baroclinic instabilities are considered for a basic flow with variable shear.[49]

Detailed investigation and comparaison of these mechanisms has still to be performed in realistic atmospheric models. There is, however, no doubt that the study of the coherent forcing due to the small scales is a key problem in the attempt to produce climatic forecasts.

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Quasi-Periodic Route to Chaos—The Case of a Fixed Winding Number

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A Rayleigh–Benard experiment in mercury[1,2] where a natural frequency (the oscillatory instability) is forced with a second frequency[3] (electromagnetic excitation) is experimentally studied. (References 1 and 2 correspond to the Rayleigh–Bernard theory. Reference 2 is very physical.) The motion is on a 2-torus in phase space, with orbits either locked on a periodic trajectory or on a quasi-periodic one on the torus. For quasi-periodic states of fixed winding number the transition to chaos, as the nonlinearities are increased, has a defined scaling[4,5,6] (which give the scaling laws for a fixed winding number). We have measured the δ value for two irrational winding numbers $(\sqrt{5}-1)/2$ and $\sqrt{2}-1$. The locked states define a fractal region[7,8] with a dimension $D=0.865$, which has been measured. (These references relate more to the locked states and their fractal dimensions.)

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Boolean Delay Equations and Climate Dynamics

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Boolean delay equations (BDEs) are evolution equations for a vector of discretely valued components $\mathbf{x}(t)$. The value of each component depends on previous values of components $x_j(t-t_{ij})$ according to the equations

$$x_i(t) = f_i[x_1(t-t_{i1}),..., x_n(t-t_{in})]$$

where the f_i are appropriate functions.

The existence, uniqueness, and continuous dependence of solutions both on initial data ($\mathbf{x}(t)$ given on $[0, 1]$) and on parameters (t_{ij}) allow the study of the piecewise constant initial data problem as a dynamical system, i.e., a homeomorphism T_f: $X \to X$ on a certain metric space X. This point of view is articulated in Ref. 1, where the initial data problem is formulated as a dynamical system. In order to obtain existence of piecewise constant solutions the authors introduce an upper bound on the asymptotic growth of a phase function, which can be taken as a measure of the complexity of a solution, and exhibit numerical evidence of a system with increasing complexity. In Ghil and Mullhaupt,[3] and in Mullhaupt,[4] these results are refined and complemented by lower bounds, some of which are sharp. Previous to Dee and Ghil,[2] other investigators considered equations related to BDEs or ultimately shown to be specializations of BDEs. Perhaps the most important of these are the algebraic and statistical work on shift register sequences (see the monograph of Golomb[5] for a good survey) and the dynamical studies of Thomas on kinetic logic (see the collection Thomas[1]).

The subject of this lecture is the consideration of periodic and aperiodic solutions of BDEs and the consequences for the application of BDEs to modeling. Relevant are the results that (1) systems with rationally dependent delays have eventually periodic solutions (2) systems with arbitrary delays can have aperiodic solutions with complexity dependent upon the resonance relations satisfied by the delays; and (3) systems are structurally stable if and only if all solutions have transients and periods that are uniformly bounded and resonance is not of low order.

A concrete example of applying BDEs to climate is taken from Ghil, Mullhaupt, and Pestiaux.[6] Two systems of BDEs are given which model the effect of abyssal circulation during ice ages. The dynamic behavior is investigated using methods of Ghil and Mullhaupt[3] and qualitative agreement is obtained for one of the two competing systems with paleoclimatic data.

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4. A. Mullhaupt, *Boolean Delay Equations: A Class of Semi-Discrete Dynamical Systems*, Ph.D. dissertation, New York University (1984). This contains some results in common with Ghil and Mullhaupt,[3] with detailed proofs. In addition, two appendices are given, one explaining various numerical algorithms for computing BDEs and their relative merits, the other a survey of diophantine approximation, which considers the question of resonances for BDEs.
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B. Related Material

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[1] How many integers of the form 2^m3^n are less than x?

High-Resolution Schemes for Gas Dynamics

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We construct nonoscillatory methods for approximating hyperbolic systems of conservation laws based on work in Refs. 1 and 2. These methods share many desirable properties with TVD-total variation diminishing schemes, but TVD schemes are at most first-order accurate at critical points of the solution.

The schemes are first developed for scalar conservation laws but they have natural extensions to systems, including gas dynamics, via Riemann solvers—either Godunov's method or one of several approximations.

TVD schemes based on similar ideas have been very successful. They have resulted in state-of-the-art calculations for problems with strong shocks, and have been incorporated into production codes at NASA laboratories to calculate transonic and supersonic flows in aeronautics.

They are reliable enough so that new physics can sometimes be discovered. It was found numerically, using a TVD scheme, that the equation of magnetohydrodynamics do not have only genuinely nonlinear linearly degenerate fields as was generally believed. The analogue of a nonconvex field sometimes exists. This was then verified analytically.[28]

The new schemes are always designed to have the following properties:

(−1) Consistency

(0) Conservation form

(1) Sharp monotone discrete shock profiles—no spurious overshoots

(2) High-order accuracy in smooth regions of the flow right up to discontinuities

(3) No nonphysical limit solutions ⇔ entropy condition

(4) Rapid convergence to steady state, when appropriate.

Conventional schemes such as Lax–Wendroff's have problems with (1) and also with (3), which can be fixed. Besides polluting the solution, the spurious wiggles can trigger nonlinear instabilities, e.g., negative densities.

Condition (3) can be very important. A simple entropy fix of the Murman scheme used to solve the transonic small disturbance equation resulted in a much more robust algorithm. Implicit calculations could be performed with a factor of thirty greater time steps, for unsteady transonic flutter problems. NASA production codes have been changed accordingly.[15,16]

We review the recent history of this subject and describe the new nonoscillatory and high-order accurate methods.

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ADDENDUM

The References 1 and 2 concern the new numerical methods. References 3–14, 18–20, and 25–27 concern TVD schemes at various stages of development, as well as practical calculations. References 15 and 16 use an entropy fix of Murman's scheme for transonic calculations first suggested in Ref. 17. References 21–24 concern the transonic full potential equation, and Ref. 28 concerns the MHD equations, including the new results mentioned above.

Intermittency in Turbulence

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In fully developed turbulence one can measure the velocity gradient on a scale ε: $\delta_\varepsilon v(x) \equiv v(x+\varepsilon) - v(x)$.

Experimentally it seems that $\langle (\delta_\varepsilon v)^n \rangle \sim \varepsilon^{\gamma(n)}$, $\gamma(n)$ being a nontrivial function of n (it is hoped, but it is not evident, that $\gamma(n)$ does not depend on the particular system but becomes a universal function of n in the zero viscosity limit). This phenomenon is called intermittency.

It was suggested that this behavior is due to the existence of singularities with different strength, concentrated on sets having different measures, one embedded into the other. This model was further elaborated in Ref. 3.

Intermittency is also a rather general phenomenon: it is present in the "response function" of a deterministic (or stochastic) dynamical system; this is the starting point for the definition of generalized Liapunov exponents.[4,5] The density (measured with a resolution ε of points of an inhomogeneous fractal) is also intermittent.[3,6,7]

Intermittency appears also in the statistical mechanics of random systems, where we have that, in the large x limit $\overline{[C(x)]^n} \sim \exp[-m(n)x]$, $C(x)$ is the correlation function and the bar denotes the average over the randomness. (See Ref. 1 for a nice recent review.)

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The Boltzmann Grad Limit for Hard Spheres

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A basic problem in the kinetic theory of gases is to prove the validity of the Boltzmann equation. As generally claimed in the textbooks, it is believed that a system of particles obeying the Newton laws approaches, in the Boltzmann–Grad limit, a stochastic and irreversible system, whose dynamics is governed by the Boltzmann equation. Nevertheless very few rigorous results are known. First, Gallavotti posed and solved the problem for a simplified model (the Lorentz model, for which the corresponding Boltzmann equation is linear).[1] For improvements and a general discussion of this limit, see Ref. 2. See Ref. 3 for a stronger convergence and for a probabilistic approach.

Lanford, in a well-known paper[4] investigated hard-sphere systems and proved the validity of the Boltzmann equations, but only for short times. Recently[5] the same result has been proved for a two-dimensional system of hard spheres, in case of small perturbations of the vacuum, for all times. My talk concerns these last two results.

It has be mentioned that, in the Lanford's situation, even a global existence theorem is not known, while in our context (also in 3D) a satisfactory existence theorem was already obtained.[6]

For the status of the existence theorems the reader is referred to Ref. 7 and references quoted therein.

I conclude with a philosophical remark. A correct mathematical proof of the Boltzmann–Grad limit is not only a matter of elegance. There are situations, e.g., the four-velocities Brodwell model, in which the formal proof of the BG limit is simply wrong.[8]

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Einstein's Relation and Contiguity

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In his fundamental work on physical Brownian motion, Einstein[1] established the identity

$$D = 2kT \cdot m \tag{1}$$

between the diffusion constant of a Brownian particle and its mobility, which is defined as average velocity per unit force ($kT =$ Boltzmann factor). It follows from the assumption that an additional force F adds a drift $m \cdot F$ to the Wiener process describing the particles motion, and that this modified process has as invariant measure $dx \cdot \exp(F \cdot x/kT)$.

Here we examine some cases of diffusion like processes in R^d, which arise as motion of a tagged particle in a system of interacting particles or in a (constant in time) random environment. All these cases have in common that there exists an invariance principle for the renormalized trajectory under consideration

$$X_\varepsilon(t) := X(t\varepsilon^{-2}), \qquad t \geqslant 0 \tag{2}$$

tends weakly to W_D, a Wiener process with diffusion matrix D. To identify D, one proceeds as follows: one imbeds the given dynamics, which is Markovian with reversible measure μ on some state space, into a family $[T_t(b)]_{t \geqslant 0}$, $b \in R^d$, so that the measure $\mu \cdot \exp(2b \cdot u)$, u: position of the tagged particle becomes reversible for $[T_t(b)]$. For physical reasons it appears meaningful, when the rescaling is done, to replace b by εb. It turns out that the law of $X_\varepsilon(\cdot)$ under $[T_t(\varepsilon b)]$, in the limit $\varepsilon \to 0$, is contiguous to the law of $X_\varepsilon(\cdot)$ under $[T_t(0)]$, the original dynamics; the limiting process is

$$W_D(t) + vt \tag{3}$$

with

$$v = D \cdot b \tag{4}$$

In this sense eqs. (3) and (4) are the desired Einstein relation. (The "mobility" in our units is equal to D, since in our form of the invariant measure $\mu \cdot \exp(2b \cdot u)$ the Boltzmann factor kT has been set equal to $\frac{1}{2}$).

Finally we examine other approaches to Einstein's relation: identities for the second moments of $X(t)$, t finite, and an attempt to define mobility by the long-time behavior of X under $[T_t(b)]$, with b fixed.

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Ergodic Theory of Chaos and Strange Attractors

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Physical and numerical experiments show that deterministic noise, or chaos, is ubiquitous. A good understanding of the onset of chaos has been achieved earlier using as a mathematical tool the geometric theory of differentiable dynamical systems. Moderately excited chaotic systems require new tools, which are provided by the *ergodic* theory of dynamical systems. This theory has reached a stage where fruitful contact and exchange with physical experiments has become widespread. The present review is an account of the main mathematical ideas and their concrete implementation in analyzing experiments. The main subjects are the theory of *dimensions* (number of excited degrees of freedom), *entropy* (production of information), and of *characteristic exponents* (describing sensitivity to initial conditions). The relations between these quantities, as well as their experimental determination, are discussed.

The systematic investigation of these quantities provides us for the first time with a reasonable understanding of dynamical systems, excited well beyond the quasiperiodic regimes. This is another step toward understanding highly turbulent fluids.

The above title and abstract are those of a review paper written jointly with J.-P. Eckmann to appear in *Rev. Mod. Phys.*

We present, with some comments, a list of general references centered on the physical aspects of the ergodic theory of differentiable dynamical systems. These referencesinclude books, conference proceedings, and reviews.

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A Solution to the Navier–Stokes Inequality with Internal Singularities

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Informal statement of theorem. Let $\varepsilon > 0$. There exists a solution to the Navier–Stokes equations

$$\frac{\partial u_i}{\partial t} = \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} - \frac{\partial p}{\partial x_i} + \Delta u_i + f_i \qquad \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} = 0 \tag{A}$$

of viscous incompressible flow in three-space with an external force f satisfying the following

$$\sum_{i=1}^{3} f_i u_i \leqslant 0 \qquad \sum_{i=1}^{3} \frac{\partial f_i}{\partial x_i} = 0 \tag{B}$$

The flow starts out (at time 0) being C^∞ with compact support and develops singularities (at time 1) on a set of Hausdorff dimension equal to $1-\varepsilon$. At these singularities the speed of the flow becomes infinite.

Formal statement of theorem. If u: $R^3 \times (a, b) \to R^n$ is a function then we set

$$\Delta u = \sum_{i=1}^{3} \frac{\partial^2 u}{\partial x_i^2} \qquad \nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3}\right)$$

$$|\nabla u|^2 = \sum_{i=1}^{n} \sum_{j=1}^{3} \left(\frac{\partial u_i}{\partial x_j}\right)^2$$

Theorem. Let $\varepsilon > 0$. There exist $S \subset R^3$ and functions u: $R^3 \times [0, \infty) \to R^3$, p: $R^3 \times [0, \infty) \to R$ with the following properties:

1. there is a compact set $K \subset R^3$ such that $u(x, t) = 0$ for all $x \notin K$
2. for fixed t, the function $u_t: R^3 \to R^3$ defined by $u_t(x) = u(x, t)$ is C^∞
3. $\displaystyle\sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i}(x, t) = 0$
4. $\displaystyle p(x, t) = \int_{R^3} \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial u_j}{\partial x_i}(y, t) \frac{\partial u_i}{\partial x_j}(y, t)(4\pi\, |x-y|)^{-1}\, dy$
5. there exists $M < \infty$ such that $\|u_t\|_2 \leqslant M$ for all t
6. $|\nabla u|^2$, $|u|^3$ and $|u|\ |p|$ are integrable,
7. if g: $R^3 \times (0, \infty) \to R$ is a C^∞ function with compact support and $g \geqslant 0$ then

$$\int_0^\infty \int_{R^3} |\nabla u|^2\, g \leqslant \int_0^\infty \int_{R^3} (2^{-1}\, |u|^2 + p)\, u \cdot \nabla g + \int_0^\infty \int_{R^3} 2^{-1}\, |u|^2 \left(\frac{\partial g}{\partial t} + \Delta g\right)$$

8. if $x \in S$ and U is a neighborhood of $(x, 1)$, then u is not essentially bounded on U
9. S is a compact set with Hausdorff dimension equal to $1-\varepsilon$

In Ref. 3 I showed that eqs. (3), (4) and (7) imply a weak form of eqs. (A) and (B). Reference 3 is a preliminary version of the theorem with $S =$ a set with only one point. Reference 4 extends this to $S =$ a Cantor set of positive Hausdorff dimension. References 1 and 2 show that the Hausdorff dimension of S cannot be more than 1.

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Hydrodynamic Limit for Systems with Many Particles: A Bibliography

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Over recent years there has been an attempt to understand the validity of the hydrodynamic description of microscopic models with many particles. The goal is to show that in a certain scaling limit—the hydrodynamic limit—the appropriate hydrodynamic equations become exact. The hydrodynamic limit is distinguished from all other limits by the fact that only space and time are scaled: on the microscale where individual particles and their motion are resolved, the interaction among particles is unchanged.

1. CLASSICAL PARTICLES

On the Euler scale the hydrodynamic limit for one-dimensional hard rods is proved in Ref. 1. Convergence of the average density fields is shown, no law of large numbers and no local equilibrium. For the same system the hydrodynamic limit on Euler and Navier–Stokes time scale is proved for the fluctuation fields in equilibrium in Ref. 2. Only the covariance is studied. The central limit theorem on the Euler scale, and its Navier–Stokes corrections, are investigated in Ref. 3.

2. CLASSICAL LATTICE SYSTEMS

On the Euler scale the hydrodynamic limit for a system of harmonic oscillators is proved in Ref. 4. They show the law of large numbers, but no local equilibrium.

3. QUANTUM SYSTEMS

The Euler equations are proved for the one-dimensional XY model and quantum hard rods (see Refs. 5 and 6). No local equilibrium is shown. On the macroscopic scale the quantum character is lost in the limit.

4. CLASSICAL PARTICLES, STOCHASTIC DYNAMICS

In these models good ergodic properties are built into the dynamics. One wants to understand how the system manages to maintain local equilibrium. Another aspect is an understanding of the qualitative behavior of the time correlations on large space-time scales—the standard problem of equilibrium statistical mechanics.

The hydrodynamic limit on the Navier–Stokes time scale is proved for simple symmetric exclusion in Ref. 7, for independent particles in Ref. 8, for diffusing hard rods in Ref. 9, and for the zero range process with rate $g(k)=1$ in Ref. 10.

In the latter two models a nonlinear diffusion equation is obtained. There is recent progress in deriving the nonlinear diffusion equation without exploiting very specific properties of the model (as in the above cases). Reference 11 deals with the Ginzburg-Landau model with conservation law (model B) at sufficiently weak interaction and Ref. 12 with the zero range process with an approximately linear rate $g(k)$. In both cases the law of large numbers and local equilibrium are established. The Ginzburg–Landau model without noise (zero temperature) are studied in Refs. 13 and 14.

Steady states for simple exclusion are discussed in Ref. 7, Ref. 15, and in Ref. 16 for zero range.

The hydrodynamic limit on the Euler scale for asymmetric simple exclusion is proved in Refs. 17–19, establising local equilibrium. Asymmetric zero range with $g(k)=1$ is studied in Ref. 20. Its behavior at a shock is investigated in Ref. 21.

A branching process from a hydrodynamic point of view is discussed in Ref. 22. Fluctuation fields are expected to converge to an infinite-dimensional Ornstein–Uhlenbeck process (c.f., e.g., Ref. 23). This has been proved in equilibrium for general, reversible zero-range processes by Brox and Rost (Ref. 24) for lattice gases (reversible exclusion processes with speed change) satisfying the gradient condition at high temperatures (Ref. 25) for interacting Brownian particles at high temperatures in Ref. 26; c.f. also Refs. 27 and 28. For Ginzburg–Landau models with conservation law at high temperatures in Ref. 29.

Fluctuations around time-dependent solutions are studied for the zero range process in Ref. 9. Related are small deviations from local equilibrium (Refs. 30 and 31).

Long-range static correlations are a different mechanism leading to hydrodynamic behavior. For the voter model this is studied in Refs. 32–37. Reference 36 is for Ginzburg–Landau models of anharmonic lattices in Ref. 37.

Equilibrium fluctuations of interacting Brownian particles with long-range forces still behave differently. A model case is studied in Ref. 38.

Two survey articles are highly recommended: Refs. 39 and 40.

I apologize for any papers overlooked. It should be mentioned that for the motion of a test (tagged) particle of an interacting particle system the hydrodynamic limit corresponds to the central limit theorem (invariance principle), scaled in such a way that the microscopic interactions remain unchanged. This is by itself a large subject, and I list only a few recent papers which may serve as a starting point (Refs. 41–45).

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